ES1004 Econometrics by Example

Lecture 11: ARCH and GARCH Models

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Gujarati textbook, second edition [chapter 15]

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Time Series Econometrics

13. stationary and nonstationary time series
14. cointegration and error correction models
15. asset price volatility: the ARCH and GARCH models
16. economic forecasting

- previous course on time series econometrics

ES1002 Lectures

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Volatility Clustering

- financial time series, such as stock prices, interest rates, foreign exchange rates, often exhibit volatility clustering
  - periods of turbulence: prices show wide swings; and
  - periods of tranquillity: there is a relative calm
- various sources of news and other economic events may have an impact on the time series pattern of asset prices
  - news can lead to various interpretations, and economic events like an oil crisis can last for some time
  - so we often observe the large positive and large negative observations in financial time series to appear in clusters

Real and Financial Impacts

- such swings in oil prices and credit crises have serious effects
- investors are concerned about the
  - rate of return on their investment
  - risk of investment and the variability or volatility of risk
- it is important to measure asset price and asset returns volatility

Measuring Volatility

- a simple measure of asset return volatility is its variance over time
- variance by itself does not capture volatility clustering
  - subtract the mean value from individual values, square the difference and divide it by the number of observations
  - a measure of unconditional variance
  - a single number of a given sample
  - does not take into account the past history (time-varying volatility)

The ARCH Model

- autoregressive conditional heteroscedasticity
- a measure that takes into account the past history (time-varying volatility)
- in time series data involving asset returns, such as returns on stocks or foreign exchange, we observe autocorrelated heteroscedasticity

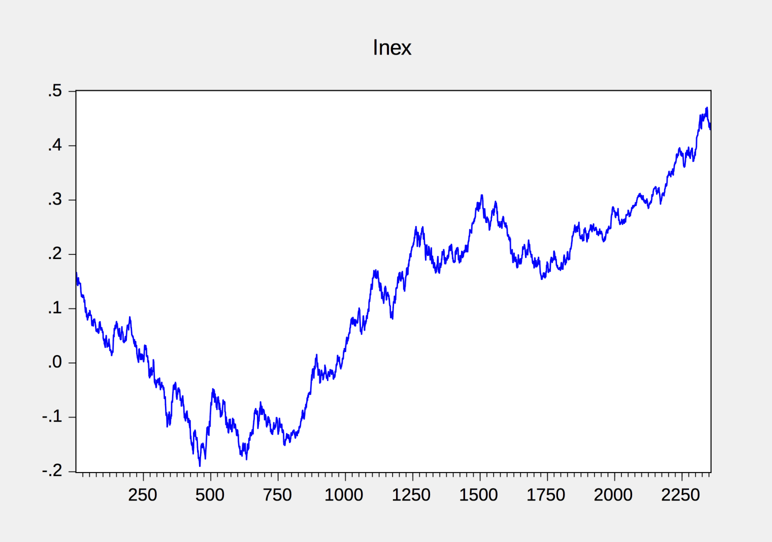
Autocorrelated Heteroscedasticity

- heteroscedasticity, or unequal variance, in cross section data because of the heterogeneity among individual cross-section units
- in time series data, we usually observe autocorrelation
- in financial data we observe autocorrelated heteroscedasticity
  - i.e., heteroscedasticity observed over different periods is autocorrelated
- in the literature, this phenomenon called ARCH effect

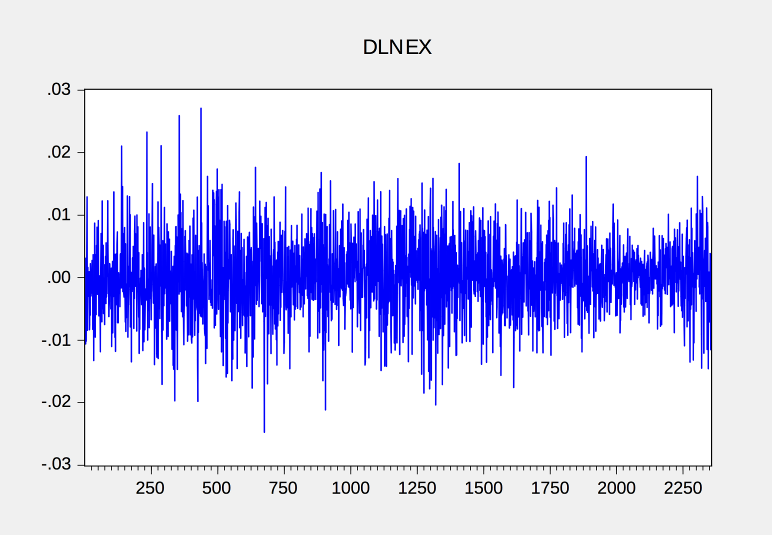
Example: Exchange Rate

- data table13_1.xls
  - the exchange rate between the us dollar and the euro EX; dollars per unit of euro
  - daily from January 4, 2000 to May 8,2008 [2355 observations]
  - are not continuous; exchange rate markets are not always open every day and because of holidays
  - see figure next slide

Exchange Rate: Log



Exchange Rate: Log-changes



Variance vs. Volatility

- the variance of a random variable is a measure of the variability in the values of the random variable
- for our data on daily exchange rate returns
  - the mean is about 0.000113 or 0.0113%
  - the variance is about 0.0000351
- this variance does not capture the volatility of the daily exchange rate return seen in previous figure
- because it does not take into account the variation in the amplitudes noticed in the figure

Measuring Volatility

- a simple way to measure volatility

$$RET_t = c + u_t$$

where RTE_t daily return, c a constant, u_t error term

- if we obtain the residuals e_t and square them, you get the plot in the next slide

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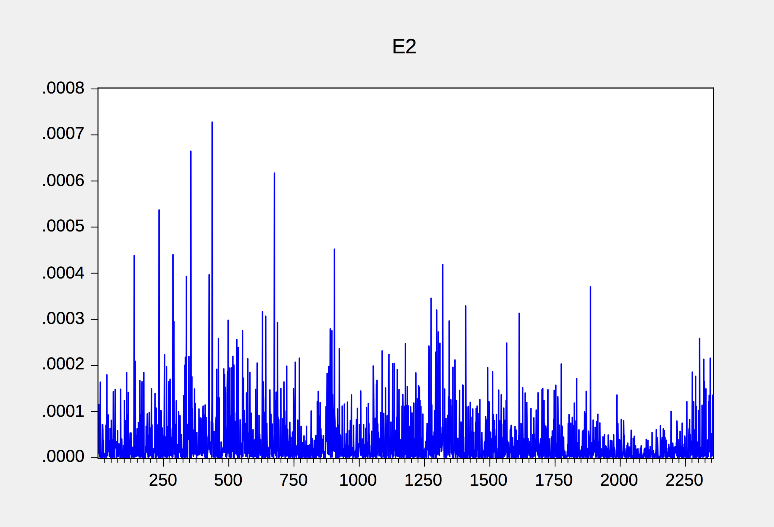
Regression

Dependent Variable: DLNEX
Method: Least Squares
Date: 10/15/16 Time: 09:23
Sample (adjusted): 2 2355
Included observations: 2354 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.000113	0.000122	0.924529	0.3553

R-squared	0.000000	Mean dependent var	0.000113
Adjusted R-squared	0.000000	S.D. dependent var	0.005926
S.E. of regression	0.005926	Akaike info criterion	-7.418381
Sum squared resid	0.082642	Schwarz criterion	-7.415932
Log likelihood	8732.434	Hannan-Quinn criter.	-7.417489
Durbin-Watson stat	1.995294		

Squared Residuals



Measuring Volatility

- wide swings in the squared residuals can be taken as an indicator for underlying volatility
- in the squared residual figure observe there
  - clusters of periods when volatility is high and clusters of periods when volatility is low
  - these clusters seems to be autocorrelated
  - when volatility is high, it continues to be high for quite some time
  - when volatility is low, it continues to be low for a while

The ARCH Model

$$Y_t | I_{t-1} = \alpha + \beta X_t + u_t$$

- Y_t exchange rate return, X_t one variable or a vector of variables

- conditional on the information available up to time $(t-1)$, the value of the random variable Y_t is a function of the variable X_t and u_t

$$u_t | I_{t-1} \sim iid \ N(0, \sigma_t^2)$$

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The ARCH Model

- in the CLARM it is assumed that $\sigma_t^2 = \sigma^2$ homoscedastic variance
- but to take into account the ARCH effect, we let

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2$$

- we assume that the error variance at time t is equal to some constant plus a constant multiplied by the squared error term in the previous time period

The ARCH Model

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2$$

- if $\lambda_1 = 0$

  - the error variance is homoscedastic

  - the framework of the CLRM applies

The ARCH Model

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2$$

- coefficients of this equation should be positive because the variance cannot be a negative number
- it is assumed that $0 < \lambda_1 < 1$

The ARCH Model

$$Y_t | I_{t-1} = \alpha + \beta X_t + u_t$$

- after taking the mathematical expectation on both sides

$$\alpha + \beta X_t \quad \text{the conditional mean equation}$$

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2 \quad \text{the conditional variance equation}$$

- both conditional on the information set I_{t-1}

ARCH(p)

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2$$

- this equation known as ARCH(1) model
  - includes only one lagged squared value of the error term
- this model can be easily extended to an ARCH(p) model, where we have p lagged squared error terms

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2 + \lambda_2 u_{t-2}^2 + \cdots + \lambda_p u_{t-p}^2$$

Testing ARCH Effect

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2 + \lambda_2 u_{t-2}^2 + \cdots + \lambda_p u_{t-p}^2$$

- if there is an ARCH effect, it can be tested by the statistical significance of the estimated λ coefficients
- if they are significantly different from zero, we can conclude that there is an ARCH effect

Estimation

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2 + \lambda_2 u_{t-2}^2 + \cdots + \lambda_p u_{t-p}^2$$

- since the u are not directly observable, we use the estimated residuals

$$\hat{u} = Y_t - \hat{\alpha}_t - \beta \hat{X}_t$$

- then we estimate the following model

$$\hat{u}_t^2 = \lambda_0 + \lambda_1 \hat{u}_{t-1}^2 + \lambda_2 \hat{u}_{t-2}^2 + \cdots + \lambda_p \hat{u}_{t-p}^2$$

The ARCH Model

$$\hat{u}_t^2 = \lambda_0 + \lambda_1 \hat{u}_{t-1}^2 + \lambda_2 \hat{u}_{t-2}^2 + \cdots + \lambda_p \hat{u}_{t-p}^2$$

- *AR* we are regressing squared residuals on its lagged values going back to p periods
- *CH* variance is conditional on the information available up to time $(t-1)$

ARCH(8) OLS Estimation

Dependent Variable: E2
Method: Least Squares
Date: 10/15/16 Time: 14:24
Sample (adjusted): 10 2355
Included observations: 2346 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	2.57E-05	2.21E-06	11.62358	0.0000
E2(-1)	-0.005920	0.020674	-0.286340	0.7746
E2(-2)	0.009899	0.020645	0.479485	0.6316
E2(-3)	0.022836	0.020603	1.108414	0.2678
E2(-4)	0.060409	0.020591	2.933721	0.0034
E2(-5)	0.037337	0.020588	1.813520	0.0699
E2(-6)	0.064005	0.020596	3.107659	0.0019
E2(-7)	0.047062	0.020631	2.281178	0.0226
E2(-8)	0.031118	0.020654	1.506680	0.1320

R-squared	0.014311	Mean dependent var	3.50E-05
Adjusted R-squared	0.010936	S.D. dependent var	5.94E-05
S.E. of regression	5.90E-05	Akaike info criterion	-16.63271
Sum squared resid	8.15E-06	Schwarz criterion	-16.61061
Log likelihood	19519.17	Hannan-Quinn criter.	-16.62466
F-statistic	4.241191	Durbin-Watson stat	1.998521
Prob(F-statistic)	0.000045		

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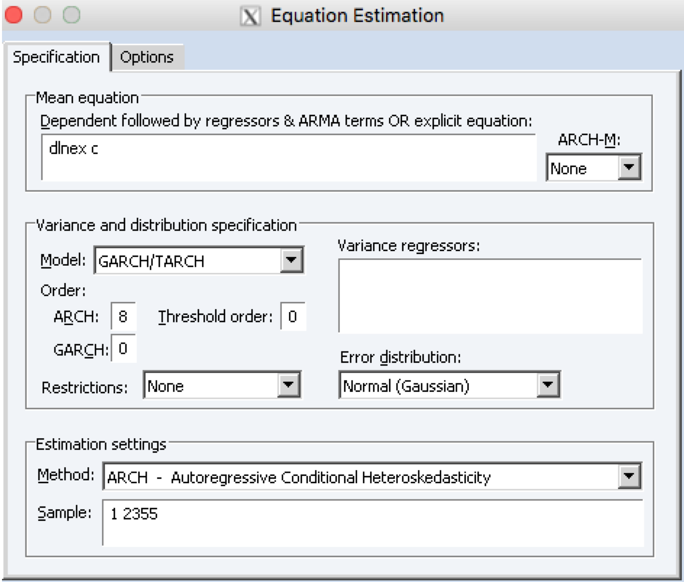
Estimation of ARCH Model

- the maximum likelihood approach
  - an advantage of the ML method is that we estimate the mean and variance functions simultaneously
  - statistical packages such as stata and eviews, have built-in routines to estimate ARCH models

ARCH(8) ML Estimation



ARCH(8) ML Estimation

Dependent Variable: DLNEX
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 10/15/16 Time: 14:48
Sample (adjusted): 2 2355
Included observations: 2354 after adjustments
Convergence achieved after 15 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-2)^2 + C(5)*RESID(-3)^2
+ C(6)*RESID(-4)^2 + C(7)*RESID(-5)^2 + C(8)*RESID(-6)^2 + C(9)
*RESID(-7)^2 + C(10)*RESID(-8)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000169	0.000116	1.461982	0.1437
Variance Equation				
C	2.16E-05	1.57E-06	13.76182	0.0000
RESID(-1)^2	0.003932	0.014396	0.273141	0.7847
RESID(-2)^2	0.016986	0.020145	0.843199	0.3991
RESID(-3)^2	0.030099	0.016475	1.827011	0.0677
RESID(-4)^2	0.058977	0.022443	2.627792	0.0086
RESID(-5)^2	0.061454	0.025199	2.438777	0.0147
RESID(-6)^2	0.088798	0.023936	3.709831	0.0002
RESID(-7)^2	0.058582	0.020295	2.886554	0.0039
RESID(-8)^2	0.076217	0.023279	3.274017	0.0011

R-squared	-0.000090	Mean dependent var	0.000113
Adjusted R-squared	-0.000090	S.D. dependent var	0.005926
S.E. of regression	0.005927	Akaike info criterion	-7.435345
Sum squared resid	0.082650	Schwarz criterion	-7.410860
Log likelihood	8761.401	Hannan-Quinn criter.	-7.426428
Durbin-Watson stat	1.995115		

ARCH(8) ML Estimation

Dependent Variable: DLNEX
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 10/15/16 Time: 14:48
Sample (adjusted): 2 2355
Included observations: 2354 after adjustments
Convergence achieved after 15 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*RESID(-2)^2 + C(5)*RESID(-3)^2
+ C(6)*RESID(-4)^2 + C(7)*RESID(-5)^2 + C(8)*RESID(-6)^2 + C(9)
*RESID(-7)^2 + C(10)*RESID(-8)^2

Variable	Coefficient	Std. Error	z-Statistic	Prob.
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Variance Equation

ARCH(8) ML Estimation

Variance Equation

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Sum squared resid	0.082650	Schwarz criterion	-7.410860
Log likelihood	8761.401	Hannan-Quinn criter.	-7.426428
Durbin-Watson stat	1.995115		

Drawbacks of ARCH Model

- requires estimation of the coefficients of p autoregressive terms, which consumes several degrees of freedom
- difficult to interpret all the coefficients, especially if some of them are negative
- the OLS estimating procedure does not lend itself to estimate the mean and variance function simultaneously
- the literature suggests that any model higher than ARCH(3) is better estimated by GARCH

GARCH Model

- generalised autoregressive conditional heteroscedasticity
- we modify the variance equation to get GARCH(1,1) as follows

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2 + \lambda_2 \sigma_{t-1}^2$$

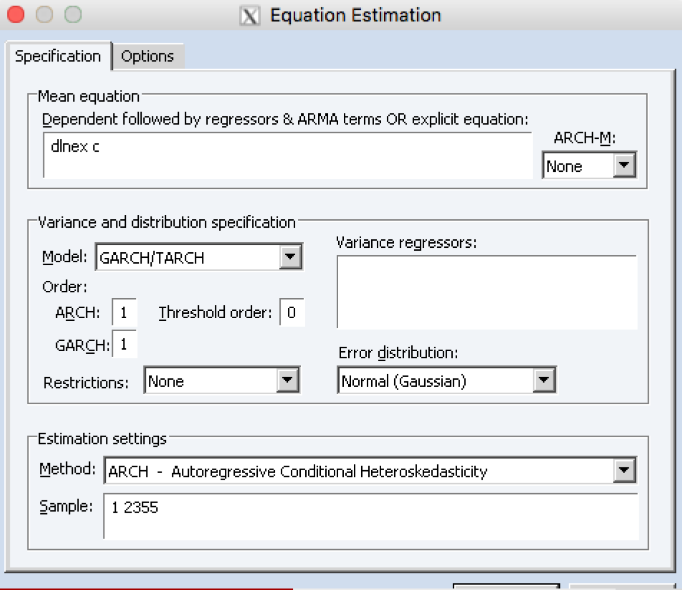
- conditional variance at time t depends on
  - the lagged squared error term at time $(t-1)$, and
  - the lagged variance term at time $(t-1)$

GARCH(1,1)

$$\sigma_t^2 = \lambda_0 + \lambda_1 u_{t-1}^2 + \lambda_2 \sigma_{t-1}^2$$

- it can be shown that ARCH(p) model is equivalent to GARCH(1,1) as p increases
- in ARCH(p) we have to estimate $(p+1)$ coefficients, whereas in GARCH(1,1) model we estimate only 3 coefficients
- GARCH(1,1) can be extended to GARCH(p,q) model
  - p lagged squared error terms
  - q lagged conditional variance terms
- in practice, GARCH(1,1) has proved useful to model returns on financial assets

GARCH(1,1)



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GARCH(1,1)

Dependent Variable: DLNEX
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 10/15/16 Time: 15:12
Sample (adjusted): 2 2355
Included observations: 2354 after adjustments
Convergence achieved after 37 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000189	0.000110	1.719603	0.0855
		Variance Equation		
C	7.92E-08	5.08E-08	1.559218	0.1189
RESID(-1)^2	0.022842	0.004086	5.590757	0.0000
GARCH(-1)	0.975189	0.004415	220.8949	0.0000

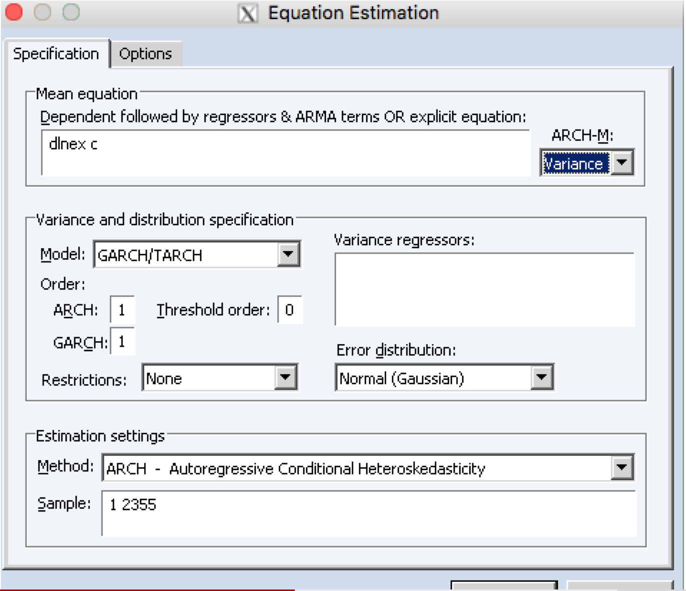
R-squared	-0.000164	Mean dependent var	0.000113
Adjusted R-squared	-0.000164	S.D. dependent var	0.005926
S.E. of regression	0.005927	Akaike info criterion	-7.473003
Sum squared resid	0.082656	Schwarz criterion	-7.463209
Log likelihood	8799.724	Hannan-Quinn criter.	-7.469436
Durbin-Watson stat	1.994967		

The GARCH-M Model

- modify the mean equation by explicitly introducing the risk factor, the conditional variance, to take into account the risk

$$Y_t = \alpha + \beta X_t + \gamma \sigma_t^2 + u_t$$

GARCH-M(1,1) Example



GARCH-M(1,1) Example

Dependent Variable: DLNEX
Method: ML ARCH - Normal distribution (BFGS / Marquardt steps)
Date: 10/15/16 Time: 15:26
Sample (adjusted): 2 2355
Included observations: 2354 after adjustments
Convergence achieved after 42 iterations
Coefficient covariance computed using outer product of gradients
Presample variance: backcast (parameter = 0.7)
GARCH = C(3) + C(4)*RESID(-1)^2 + C(5)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
GARCH	-19.30676	9.607256	-2.009602	0.0445
C	0.000780	0.000316	2.466145	0.0137
Variance Equation				
C	8.07E-08	4.96E-08	1.627226	0.1037
RESID(-1)^2	0.022576	0.003982	5.668816	0.0000
GARCH(-1)	0.975357	0.004321	225.6993	0.0000

R-squared	0.001562	Mean dependent var	0.000113
Adjusted R-squared	0.001138	S.D. dependent var	0.005926
S.E. of regression	0.005923	Akaike info criterion	-7.474165
Sum squared resid	0.082513	Schwarz criterion	-7.461922
Log likelihood	8802.092	Hannan-Quinn criter.	-7.469706
Durbin-Watson stat	1.998161		

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